# Mini-course: Fock-Goncharov coordinates and spectral networks

## Chapter 1: Teichmüller space for non-closed surfaces

For the whole chapter, $S = S_{g,h}$ will denote a surface of genus $g \geq 0$ with $h \geq 1$ points removed (called holes), such that $\chi(S) = 2-2g-h < 0$

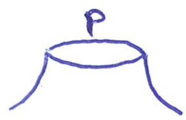
### I Hyperbolic structures on S

**Def:** The Teichmüller space of $S_{g,h}$ is:

$\mathcal{T}(S) = \left\{ (M, f) \;\middle|\; \begin{array}{l} M \text{ is a finite area (not necessarily complete) hyperbolic surface,} \\ f: M \to S \text{ is an orientation preserving homeo} \end{array} \right\} \Big/ \text{isotopy}$

When $S$ is endowed with a hyperbolic structure, we can define for all hole $p$ its width as the infimum of length of simple closed curves around it. We denote that width by $\ell(p)$.

Two possible types:

* $\ell(p) > 0$: $p$ is a "funnel"  , the metric completion of $S$ turns $p$ into a geodesic boundary component



* $\ell(p) = 0$: $p$ is a "cusp" 



**Rk:** The universal cover of $S$ is a convex open subset of $\mathbb{H}^2$, and $S \simeq \mathbb{H}^2/\Gamma$ where $\Gamma \subseteq PSL_2(\mathbb{R}) = \mathrm{Isom}^+(\mathbb{H}^2)$ is a subgroup isomorphic to $\pi_1(S)$

The map $\pi_1(S) \to PSL_2(\mathbb{R})$ is called the holonomy representation of $S$, unique up to conjugation:

$$\mathcal{T}(S_{g,h}) \subseteq \mathrm{Hom}(\pi_1 S, PSL_2(\mathbb{R}))/PSL_2(\mathbb{R})$$

is the subset of discrete and faithful representations.

We want to define coordinates on $\mathcal{T}(S)$. For this, we will first want to extend the definition of $\mathcal{T}(S)$. Two main reasons:

1) We want to take geodesic representatives of curves ending on holes, when the hole is a funnel, two choices:

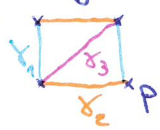
2) We will later associate to each hole a line in $\mathbb{R}^2$ stabilized by the holonomy around that hole. For funnel, two choices as holonomy is loxodromic.

## II Framed Teichmuller space and shear coordinates

Def: The framed Teichmüller space of $S$ is the space of marked hyperbolic stuctures on $S$ with a choice of orientation of each funnel, up to isotopy. We denote it by $\mathcal{T}^{x}(S)$.

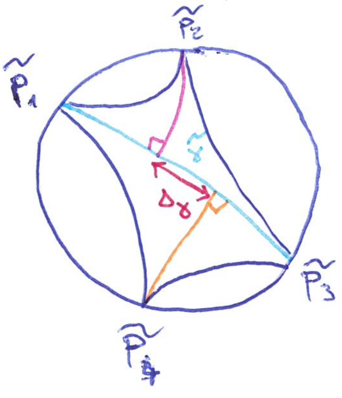
Rk: There is a natural map $\mathcal{T}^{x}(S) \to \mathcal{T}(S)$ forgetting the orientations. This map is a ramified covering.

Def: An ideal triangulation of $S$ is a set $\Delta = \{\gamma_i\}_{i \in I}$ of not self-intersecting pairwise not intersecting, non-isotopic curves in $S$ with endpoints in the set of holes of $S$, up to isotopy relative to endpoints.

Ex: Once punctured torus: 


Rk: We can "pull tight" an ideal triangulation, ie make it geodesic, the orientation around wich edges will coil around funnels being given by the framing.

We can now define the shear coordinates:
For any edge $\gamma \in \Delta$ with endpoints $p_1, p_3$, it is part of two triangles: $t_1 = (p_1, p_2, p_3)$ and $t_2 = (p_3, p_4, p_1)$. Lift all these to the universal cover $\tilde{S} \subset \mathbb{H}^2$:



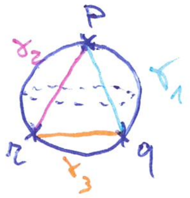

Draw the orthogonal geodesic from $\tilde{p}_2$ and $\tilde{p}_4$ to $\tilde{\gamma}$. The shear of $\gamma$ is the distance between the intersections with $\tilde{\gamma}$.
We count it positively if the first of the two orthogonal geodesic to $\tilde{\gamma}$ is on its left, negatively otherwise.

Th: The map
$$\mathcal{T}^{x}(S) \longrightarrow \mathbb{R}^{|\Delta|}$$
$$(M, f) \longmapsto \{\Delta_\gamma\}_{\gamma \in \Delta}$$
is a homeomorphism

**Prop:** Let $p$ be a hole of $S$, and let $\gamma_1, \dots, \gamma_k \in \Delta$ be the set of edges which end at $p$. Then:

* If $s_{\gamma_1}+\dots+s_{\gamma_k}>0$, $p$ is a funnel with boundary oriented according to $S$.
* If $s_{\gamma_1}+\dots+s_{\gamma_k}=0$, $p$ is a cusp
* If $s_{\gamma_1}+\dots+s_{\gamma_k}<0$, $p$ is a funnel with boundary oriented opposite to $S$.

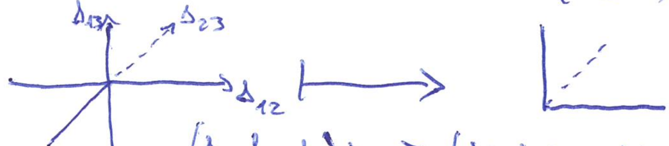
**Ex:** Pair of pants / three holed sphere:




$s_i := s_{\gamma_i}$

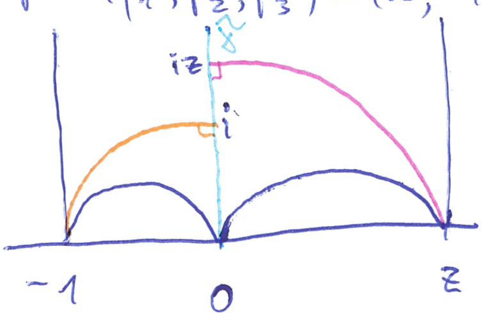
$s_{ij} := s_i + s_j$

$\mathcal{T}^\times(S) \simeq \mathbb{R}^3 \longrightarrow \mathcal{T}(S)$




$(s_{12}, s_{23}, s_{13}) \longmapsto (|s_{12}|, |s_{23}|, |s_{13}|)$

**Rk:** For later purpose, let us reformulate the construction of the shear coordinates: Up to $PSL_2(\mathbb{R})$ action, we can fix $(\tilde{p}_1, \tilde{p}_2, \tilde{p}_3) = (\infty, -1, 0)$, then $\tilde{p}_3$ is sent to $z = [\tilde{p}_1; \tilde{p}_2; \tilde{p}_3; \tilde{p}_4]$




So $s_\gamma = \log z$, i.e. $[\tilde{p}_1; \tilde{p}_2; \tilde{p}_3; \tilde{p}_4] = e^{s_\gamma}$

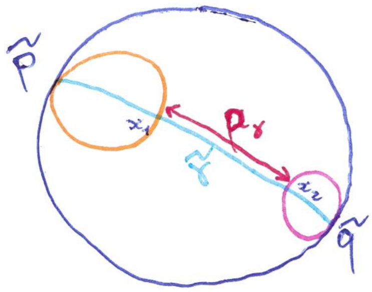
# III Decorated Teichmüller space and Penner coordinates

We want to study more carefully the part of $\mathcal{T}(S)$ where all holes are punctures. For this, we further refine the moduli space:

**Def:** The decorated Teichmüller space of $S$ is the set of hyperbolic structures for which all holes are punctures, together with the choice of a horocycle at each puncture, up to isotopy. We denote it by $\mathcal{T}^A(S)$.

**Rk:** The choice of an horocycle is equivalent to the choice of a vector in the only line of $\mathbb{R}P^1$ stabilized by the monodromy around the puncture.

We once again fix a triangulation $\Delta$ and define the Penner coordinate associated to $\gamma \in \Delta$:




Lift $\gamma$ to the universal cover. The two horocycles given by the decoration at the endpoints of $\gamma$ intersect $\tilde{\gamma}$ in $x_1, x_2$ respectively. Then $\rho_\gamma := d(x_1, x_2)$, positively if the horocycles don't cross, negatively otherwise

**Theorem:** The map $\mathcal{T}^A(S) \longrightarrow \mathbb{R}^{|\Delta|}$ is a homeomorphism.
$$(m, f) \longmapsto \{\rho_\gamma\}_{\gamma \in \Delta}$$

**Rk:** If the two horocycles correspond to vectors $v_1, v_2 \in \mathbb{R}^2$, then $\det(v_1, v_2) = e^{\rho_\gamma}$.

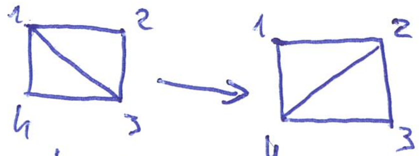
# Chapter 2 : Fock-Goncharov coordinates

The goal of this chapter is to understand how shear/Penner coordinates change with different choices of triangulation, and to generalize the whole picture to $SL_n(\mathbb{R})/PGL_n(\mathbb{R})$.

## I Combinatorial setup

The first step is to understand the combinatorics of ideal triangulations. This is mainly done through the following result:

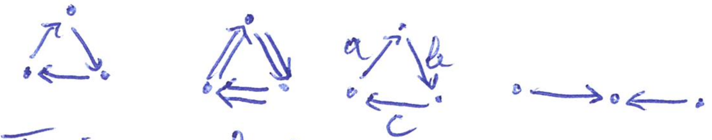
**Th:** Let $\Delta$ and $\Delta'$ be two ideal triangulation on $S$. Then there exists a finite sequence of flips turning $\Delta$ into $\Delta'$.

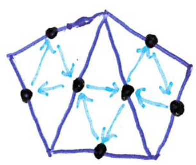
Moves of the form 


This means that we just need to understand the change of coordinates induced by a flip!

Another useful generalization (for $SL_n/PGL_n$), is the reformulation in terms of quivers and mutations:

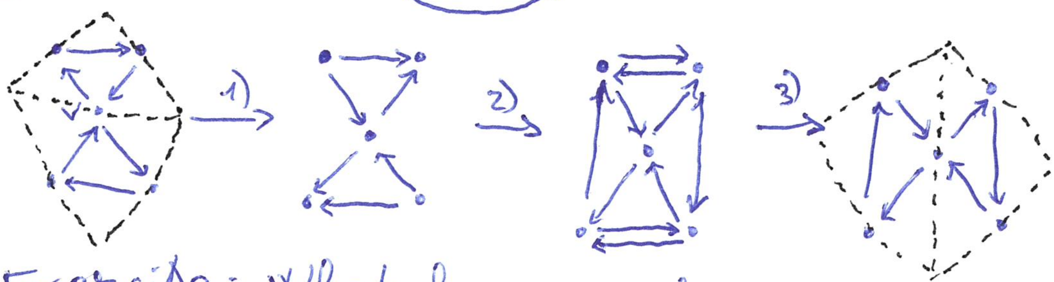
**Def:** A quiver is an oriented finite graph without 1- or 2-cycles. the edges are counted with multiplicity

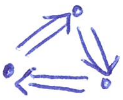
**Ex:** 


Triangulations to quivers: 

**Def:** Let $Q$ be a quiver and $v \in Q$ a vertex. The mutation at $v$ of $Q$ is the new quiver $\mu_v(Q)$ obtained by:
1) Reverse all arrows incident to $v$
2) complete all pairs $i \to v \to j$ wit a new edge $j \to i$
3) Remove created 2-cycles

**Ex:** $\bullet \longrightarrow \overset{v}{\bullet} \longrightarrow \bullet \xrightarrow{\mu_v}$ $\begin{tikzcd} \bullet & \bullet \arrow[l] & \bullet \arrow[l] \arrow[ll, bend left] \end{tikzcd}$




Exercise: What happens when mutating  at any vertex?

## II Coordinates of type $\mathcal{X}$.

**Def:** A (full) flag in $\mathbb{R}^n$ is a sequence $0 \subseteq F_1 \subseteq F_2 \subseteq \dots \subset F_{n-1} \subseteq \mathbb{R}^n$ of subspaces such that $\dim F_i = 1$. The space of flags of $\mathbb{R}^n$ is denoted by $\mathcal{F}(\mathbb{R}^n)$.

**Ex:** $\mathcal{F}(\mathbb{R}^2) = \mathbb{R}P^1$

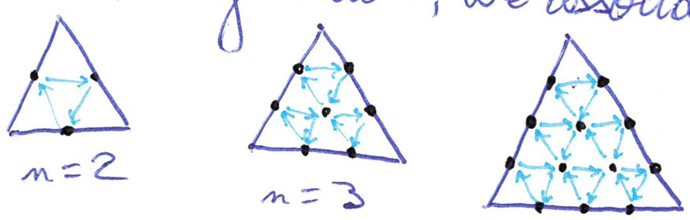
**Def:** The $\mathcal{X}$-space of $S$ is:

$$\mathcal{X}_S = \left\{ (\rho, (F^{(p)})_{p \in \mathrm{Holes}(S)}) \;\middle|\; \begin{array}{l} \rho \in \mathrm{Hom}(\pi_1 S, PGL_n(\mathbb{R})) \\ F^{(p)} \text{ is a } \rho(\underset{\text{small loop around } p}{\gamma_p})\text{-invariant flag} \end{array} \right\} \Big/ PGL_n(\mathbb{R})$$

**Rk:** When $n=2$ (up to changing $PGL_2$ to $PSL_2$, minor detail), we have naturally $\mathcal{T}^{\times}(S) \hookrightarrow \mathcal{X}_S$

**Rk:** Each triangulation of $S$ will map an open dense set of $\mathcal{X}_S$.

To a triangulation, we associate the following quivers:



$n=2$ $\quad$ $n=3$ $\quad$ $n=4$ $\quad$ - - -

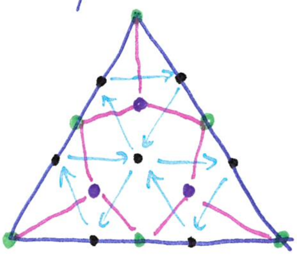
We will associate to each vertex one coordinate

First let us give an alternate definition for the cross ratio:

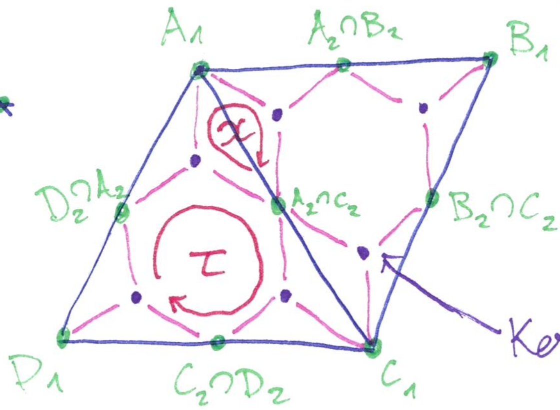
**Prop:** Let $A, B, C, D$ be 4 distinct lines in $\mathbb{R}^2$. Let $K_1 = \mathrm{Ker}(A \oplus B \oplus C \longrightarrow \mathbb{R}^2$, $a, b, c \longmapsto a+b+c)$ $\quad K_2 = \mathrm{Ker}(C \oplus D \oplus A \to \mathbb{R}^2)$.

Then the monodromy around $\begin{tikzcd} & K_1 \arrow[dr] & \\ A \arrow[ur] & & C \arrow[dl] \\ & K_2 \arrow[ul] & \end{tikzcd}$ is $[A, B, C, D]$

Let us construct a graph whose faces are the vertices of the quiver:



When 3 flags are chosen, we can endow vertices with lines and edges with maps:




$$\mathrm{Ker}(C_1 \oplus A_2\cap C_2 \oplus B_2\cap C_2 \longrightarrow C_2)$$

$$\downarrow \qquad \downarrow \qquad \downarrow \quad \text{natural projection}$$

$$C_1 \qquad A_2\cap C_2 \qquad B_2\cap C_2$$

The monodromies around the faces are either cross ratios or something new: it is called a triple ratio.
We assign those real numbers to the vertices of the quiver.

**Prop:** Mutating a vertex $i$ of the quiver has the following effect on coordinates:

$$\begin{cases} \mu_i(x_i) = x_i^{-1} & \\ \mu_i(x_j) = x_j & \text{if no edge between } i \text{ and } j \\ \mu_i(x_j) = x_j(1+x_i) & \text{if } i \to j \\ \mu_i(x_j) = x_j(1+x_i^{-1})^{-1} & \text{if } i \leftarrow j \end{cases}$$

**Rk:** No $-$ sign in the formulas: if all numbers are positive in a given triangulation, they are positive in every triangulation. The locus of $\mathcal{X}_S$ with all positive coordinates is denoted by $\mathcal{X}_S^+$.

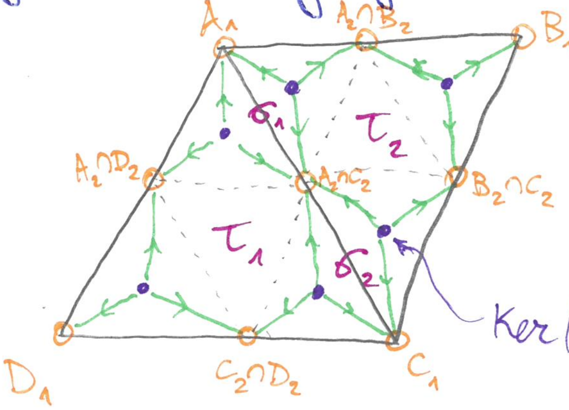
**Th:** Any representation in $\mathcal{X}_S^+$ is discrete and faithful, and $\mathcal{X}_S^+$ is a connected component of $\mathcal{X}_S$.

**Th:** When $n=2$, $\mathcal{X}_S^+ = \mathcal{T}^*(S)$

# Chapter 3: Spectral networks

## I From $\mathcal{X}$-coordinates to a line bundle.

Recall the following construction given a triple/quadruple of transverse flags $(A,B,C,D)$ in $\mathbb{R}^n$ $(n=3)$ (more generally on a rk $n$ framed vector bdle)




$\sigma_i, \tau_i$ : cross/triple ratios, monodromy around the cell

o : Associated to lines of the flags

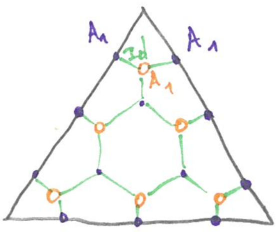
• : Associated to planar relations

$$\mathrm{Ker}(C_1 \oplus A_2\cap C_2 \oplus B_2\cap C_2 \to C_2)$$

$$\begin{array}{ccc} \downarrow & \downarrow & \downarrow \\ C_1 & A_2\cap C_2 & B_2\cap C_2 \end{array}$$

maps (isomorphisms by transversality)

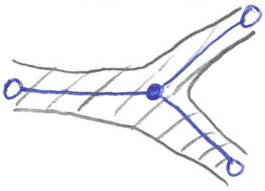
We slightly change the graph to avoid having vertices at punctures:




Gluing $\rightsquigarrow$ all triangles  Bipartite graph $\Gamma_n$ embedded on $S$ (triangulated ~~dotted~~ surface)
= ribbon graph

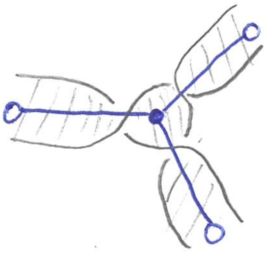
Two ways of constructing a surface out of a ribbon graph:

(1)



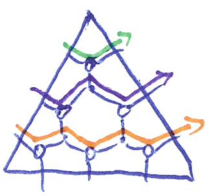
$\rightsquigarrow$ reconstructs $S$

(2)



$\rightsquigarrow$ Give rise to another surface $\Sigma$

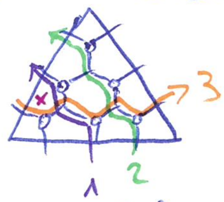
Alternate description of $\Sigma$: Glue on $\Gamma$ a punctured disk on each "zig-zag" paths:

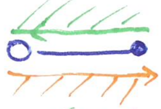


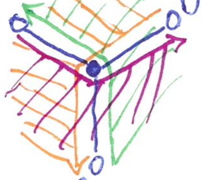
Let's detail the topology of $\Sigma$ (for our specific graph $\Gamma_n$):

* Each point of $S\setminus\Gamma_n$ is on the left of exactly $n$ zig-zag paths $\rightsquigarrow$ There is a map $\pi: \Sigma \to S$ that is $n:1$ on $S\setminus\Gamma_n$!

Ex:




* The neighborhood of an edge is  $\rightsquigarrow$ $\Sigma$ is smooth around edges
* The neighborhood of a white vertex is  $\rightsquigarrow$ $\Sigma$ is smooth around white vertices
* The neighborhood of a trivalent black vertex is:



$\rightsquigarrow$ $\Sigma \to S$ is locally like $\begin{matrix} \mathbb{D} \to \mathbb{D} \\ z \mapsto z^2 \end{matrix}$ around black vertices

<u>Conclusion</u>: $\pi: \Sigma \to S$ is a simply ramified $n:1$ cover, with simple ramification points at every trivalent black vertices.

The line bundle on $\Gamma_n$ extend to a flat line bundle on $\Sigma$, called the abelianization of the rank $n$ flat bundle on $S$ (the one from which we got the $\mathcal{X}$-coordinates).

$\to$ Easily constructed from the $\mathcal{X}$ coordinates! But how to recover the rk $n$ bundle on $S$, and the framing?

## II Spectral networks and non-abelianization

We have a flat line bundle $\mathcal{L}$ on $\Sigma$, we can push it forward to $S$: $\mathcal{E}_x = \bigoplus_{x_i \in \pi^{-1}(x)} \mathcal{L}_{x_i}$

This defines a rank $n$ vector bundle on $S\setminus\underbrace{\{\text{ram pts}\}}_{\mathcal{B}}$.

What about the flat connection?

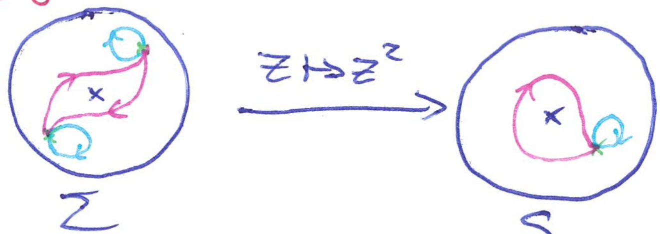
Given $\gamma: \underset{\in S\setminus\mathcal{B}}{x} \longrightarrow \underset{\in S\setminus\mathcal{B}}{y}$, we want a map

$$\nabla_\gamma : \mathcal{E}_x = \bigoplus_{x_i \in \pi^{-1}(x)} \mathcal{L}_{x_i} \longrightarrow \bigoplus_{y_j \in \pi^{-1}(y)} \mathcal{L}_{y_j} = \mathcal{E}_y$$

We can lift $\gamma$ to $\gamma_1 \dots \gamma_n$ on $\Sigma$ and define $\nabla_\gamma = \nabla_{\gamma_1} \oplus \dots \oplus \nabla_{\gamma_n}$.

⚠ No! Path lifting to ramified coverings is <u>not</u> homotopy-invariant.

Ex:

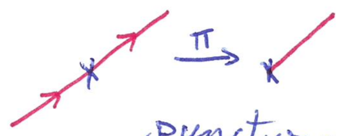



The blue and pink paths are homotopic in $S$ but their lifts aren't in $\Sigma$.

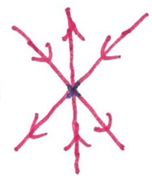
<u>Solution</u>: Fix the path-lifting map.

<u>Def</u>: A small spectral network $\mathcal{W}$ on $\Sigma$ is a set of lines $\alpha : (-1,1) \to \Sigma$ s.t.:

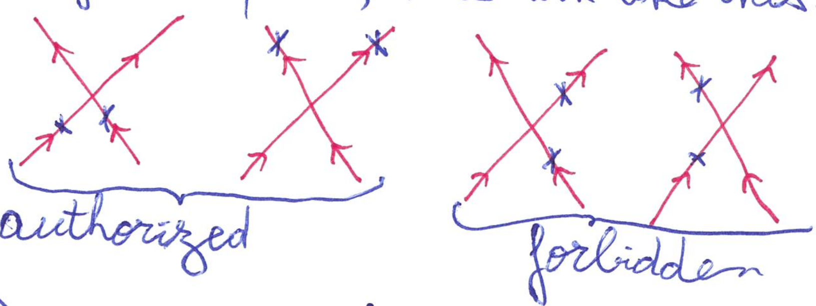
* $\forall t \in (-1,1)$, $\pi(\alpha(t)) = \pi(\alpha(-t))$



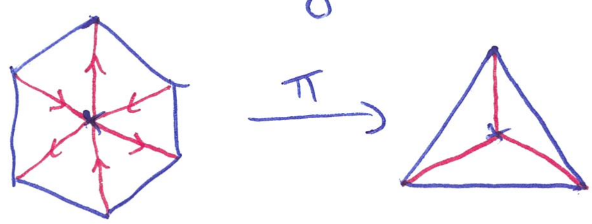

* $\alpha(t) \xrightarrow[t \to 1]{} p_1$, $\alpha(t) \xrightarrow[t \to -1]{} p_2$, $\{p_1, p_2\} \subset \pi^{-1}(p)$ ← puncture
* $\alpha(0) \in B$ is the only ramification point on $\alpha$.
* Around every ramification point, 3 lines look like this:



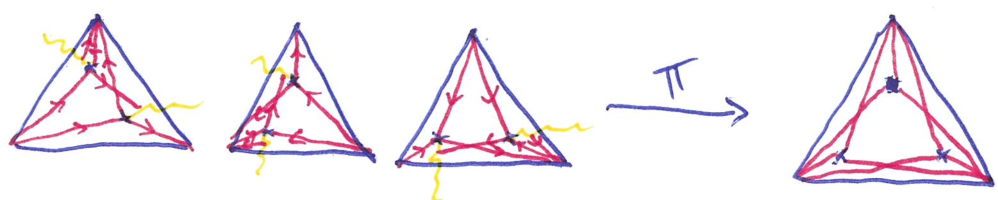
* Intersections:




<u>Ex</u>: Rank 2:




Rank 3:




<u>Def</u>: The twisted path algebra of $\Sigma$ is:

$$TPA(\Sigma) = \mathbb{Z}[\mathrm{Path}(\Sigma)] / I \quad , \text{ where } I = (\omega_x^b + \bar{x} \mid x \in \Sigma)$$

constant path at $x$

formal sum, multiplication given by concatenation (zero if endpoints don't match)

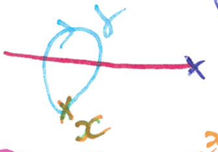
## Spectral path lifting:


The spectral lift of $\gamma$ is $SN_w(\gamma) = \gamma_1 + \dots + \gamma_n + \gamma' \in TPA(\Sigma)$

If $\gamma$ intersect several lines of the spectral network, we write $\gamma = \gamma^{(1)} \dots \gamma^{(k)}$ where each $\gamma^{(i)}$ intersect only 1 line, and $SN_w(\gamma) = SN_w(\gamma^{(1)}) \dots SN_w(\gamma^{(k)}) \in TPA(\Sigma)$

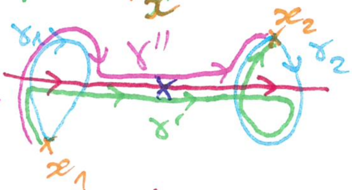
**Th:** The map $SN_w : \pi_1 S \longrightarrow TPA(\Sigma)$ is well defined, i.e. $SN_w(\gamma)$ does only depend on the homotopy class of $\gamma$.

**Proof:** lemma 1:




$SN_w(\gamma) = SN_w(\overline{x})$

Proof:

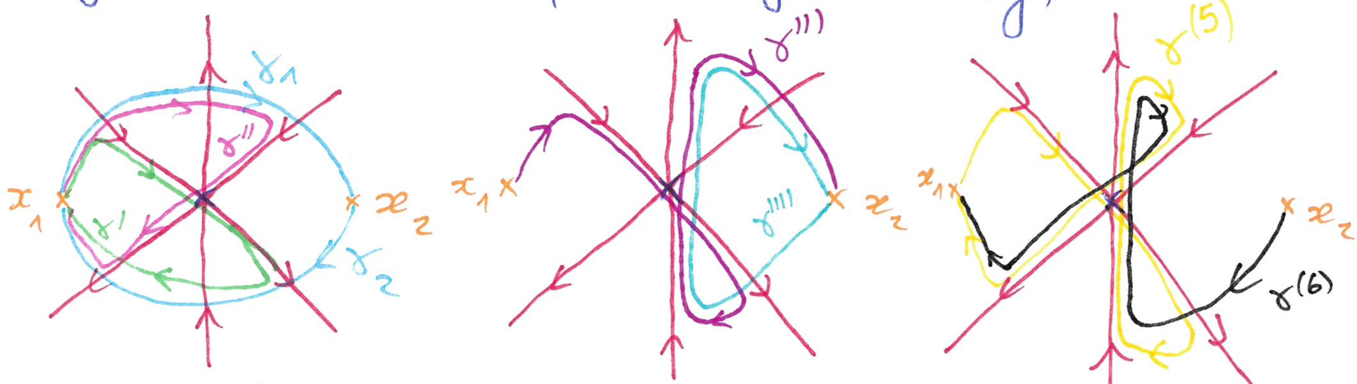



$SN_w(\gamma) = \gamma_1 + \gamma_2 + \underbrace{\gamma' + \gamma''}_{\in I} = \gamma_1 + \gamma_2 \sim \overline{x}_1 + \overline{x}_2$

lemma 2:


$SN_w(\gamma) = SN_w(\overline{x})$

Proof: (Drawn in 3 pictures for clarity)




In $TPA(\Sigma)$, $\gamma_1 + \gamma''' = 0$, $\gamma_2 + \gamma^{(6)} = 0$, $\gamma^{(5)} + \gamma' = 0$

So $SN_w(\gamma) = \gamma'' + \gamma'''' \sim \overline{x}_1 + \overline{x}_2$

We can now reconstruct the flat connection $\nabla$ on $\mathcal{E}$:
$\gamma : x \to y$, $\pi^{-1}(x) = \{x_1, \dots x_n\}$, $\pi^{-1}(y) = \{y_1, \dots y_n\}$.
In $SN_w(\gamma)$, let $\gamma_{ij}$ be the sum of paths from $x_i$ to $y_j$.
Then $\nabla_\gamma = \begin{pmatrix} \nabla_{\gamma_{11}} & \cdots & \nabla_{\gamma_{1n}} \\ \vdots & & \vdots \\ \nabla_{\gamma_{n1}} & \cdots & \nabla_{\gamma_{nn}} \end{pmatrix}$.

The flat rank $n$ bundle $n$ has a natural framing given by $\mathcal{L}_{x_1} \subset \mathcal{L}_{x_1} \oplus \mathcal{L}_{x_2} \subset \dots$ for $x$ in the neighborhood of a puncture, because the SN lift of a path around a puncture is upper-triangular, hence preserve the flag. The resulting framed rank $n$ flat vector bundle $\mathcal{E}$ is called the non-abelianization of $\mathcal{L}$.

Rk: The line bundle $\mathcal{L}$ has to have monodromy $-1$ around ramification points for the non-abelianization procedure, which is not the case of the one we constructed in the first section... There is a closely related construction of $\mathcal{L}$ that result in a bundle with monodromy $-1$ around ramification points, but this is not something I want to expand on in this talk. This is closely related to the necessity to consider twisted line configurations for Fock-Goncharov $\mathcal{X}$-coordinates.